Making choices with a binary relation: 
Relative choice axioms and transitive closures

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Abstract: This article presents an axiomatic analysis of the best choice decision problem from a reflexive crisp binary relation on a finite set (a digraph). With respect to a transitive digraph, optimality and maximality are usually accepted as the best fitted choice axioms to the intuitive notion of best choice. However, beyond transitivity (resp. acyclicity), optimality and maximality can characterise distinct choice sets (resp. empty sets). Accordingly, different and rather unsatisfying concepts have appeared, such as von Neumann-Morgenstern domination, weak transitive closure and kernels. Here, we investigate a new family of eight choice axioms for digraphs: relative choice axioms. Within choice theory, these axioms generalise top-cycle for tournaments, GOCHA, GETCHA and rational top-cycle for complete digraphs. We present their main properties such as existence, uniqueness, idempotence, internal structure, and cross comparison. We then show their strong relationship with optimality and maximality when the latter are not empty. Otherwise, these axioms identify a non-empty choice set and underline conflicts between chosen elements in strict preference circuits. Finally, we exploit the close link between this family and transitive closures to compute choice sets in linear time, followed by a relevant practical application.

Keywords: Multiple criteria analysis, Social choice theory, Digraph, Best choice, Top cycle

1. Introduction

We investigate the axiomatic foundations (Sen, 1970; Thomson, 1997; Roy and Bouyssou, 1993) of the “best” choice problem described by the following sentence: Given a non-empty finite set of alternatives and a reflexive binary relation on this set, we search for a non-empty subset containing the “best” alternatives.

Relevance of the choice problem: Decision-based fields of science are often interested in this problem. Some examples include multiple criteria decision aiding (MCDA) with the outranking methods (Roy and Bouyssou, 1993; Aït Younes et al., 2002; Bouyssou and Pirlot, 2009), the multiple attribute utility theory (Keeney and Raiffa, 1976; Figueira et al., 2005) and the decision rule approach (Greco et al., 2004, 2008; Fortemps et al., 2008; Figueira et al., 2009); economics as social choice theory and game theory (von Neumann and Morgenstern, 1944; Luce, 1956; Fishburn, 1970; Sen, 1970, 1986, 2002; Allingham, 2002; Aizerman and Aleskerov, 1995); and more recently artificial intelligence as computational social choice (Boutilier et al., 2004; Joseph et al., 2007; Lang et al., 2007; Lang and Xia, 2009; Brandt et al., 2009; Hudry, 2009) with applications in multi-agent systems and e-voting in modern societies.

For MCDA, the “best” choice problem corresponds to the exploitation problem of a crisp outranking relation in multiple criteria aiding procedures (MCAP) based on choice problematic (Roy and Bouyssou, 1993; Guitouni and Martel, 1998; Figueira et al., 2005). The subset of “best” alternatives, also called the choice set, are concretely defined in several ways: (a) the optimal set is the set of alternatives that are at least as good as all other alternatives; (b) the maximal set is the set of alternatives to which no other alternative is strictly preferred; and (c) the dominant kernel is any subset K of pairwise...
incomparable alternatives such that every alternative not in K is at most as good as one alternative of K.

To understand these choice sets, let us consider the example of choosing a postal parcels sorting machine between 9 options (see Roy and Bouyssou, 1993, Ch. 8, for details). The aggregated global opinion of the stakeholders on these options is expressed by a binary relation given in Fig. 9 (a). The optimal set is empty, but the maximal set is \{1, 2, 3, 7, 9\} and the dominant kernel is either \{1, 2, 7, 9\} or \{1, 3, 7, 9\}.

Binary relations usually contain circuits, which model the inconsistency (conflicts) in the aggregated preferences. However, when circuits are present, the three choice sets mentioned above fail to satisfy their purposes: None of them are guaranteed to identify a non-empty choice set; The dominant kernel may be non-unique and computationally intractable; They may be too large (e.g. the maximal set, see Peris and Subiza, 2002) or intuitively incoherent in real-world applications. Roy and Bouyssou (1993) note that three-quarters of the time, additional time-consuming interactions between the decision-maker and the decision support system are needed to adjust a binary relation and reveal intuitive, useful, and nonempty “best” subsets (see also Aït Younes et al., 2002). But do more appropriate choice sets exist? And if so, how may we identify them?

In economics, a burgeoning variety of new choice axioms\(^1\) has emerged to exploit chaotic preferences and highlight attractive global behaviours\(^2\): top cycle, GOCHA, GETCHA, rational top cycle, uncovered set, Copeland set, covering set, and so on (Laffond et al., 1995; Subiza and Peris, 2000, 2005b; Peris and Subiza, 1999, 2002; Kaymak and Sanver, 2003; Hudry, 2009). All these axioms assume that the binary relations are either complete (weak tournaments) or complete and antisymmetric (tournaments), because economic models rarely involve partial binary relations. Artificial intelligence, on the other hand, has opened up on incompletely specified preferences. Especially interesting in this regard are the recent computational investigations of Brandt et al. (2009). These authors have generalized six of these choice axioms (Copeland set, GOCHA, GETCHA, dominant kernel, Banks set and Slater set) to binary relations, namely one generalization per axiom.

**Our main results:** This study generalizes the top-cycle, GOCHA, GETCHA and rational top-cycle choice axioms into eight axioms on binary relations, which we refer to as relative choice axioms. These new axioms appear to produce choice sets with more acceptable properties (i.e. they are always unique, non-empty, computationally tractable, and coincide with usual choices). Pursuing the postal parcels sorting machine example, some of the most appealing relative choice axioms identify the intersection of the two dominating kernels \{1, 7, 9\} as the best choice set (we will examine this result more closely in § 5.3).

We investigate and compare the relative choice axioms by using set theory (Laffond et al., 1995; Brandt et al., 2009): Given any two choice axioms, only one of the following three cases holds: 1) the choice set characterised by one axiom always contains the choice set characterized by the other; 2) case 1 is not satisfied, but the two choice sets always intersect; or 3) for some binary relations, the two choice sets have an empty intersection. We will next show that each minimal relative choice axiom is related to a corresponding type of transitive closure of the original relation. We also check their idempotence, i.e., an axiom’s capacity to conserve choice over multiple applications of

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\(^1\) Choice axioms formally describe and characterise choice sets.

\(^2\) Although it was introduced in game theory (von Neumann and Morgenstern, 1944), the dominant kernel has been extensively investigated and generalised (e.g. quasi-kernels, semi-kernels, \((k, l)\)-kernels) in the context of graph theory (Berge, 1973; Bang-Jensen and Gutin, 2001; Ghoshal et al., 1998).
itself. Finally, we show that the relative choice axioms are efficiently computable, and then algorithmically disqualify axioms using independence (such as dominant kernels).

Outline of the paper: In section 2.1, we introduce digraphs, crisp binary preference relations, and choice sets. Next, we review the family of usual choice axioms containing optimality and maximality (§ 2.2). In section 3, we describe the four central relative choice axioms, as well as the axioms which can be defined from them combined with axiomatic components (§ 3.1). After considering their basic properties as uniqueness and idempotence, we present a literature review (§ 3.2) and we characterise the internal properties of relative choice sets (§ 3.3). In section 4, we make a set-theoretical comparison of relative choice axioms (§ 4.1). Next, we show cases when relative choice axioms and usual choice axioms coincide (§ 4.2), and point out an interesting combination of relative choice sets both conserving coincidence with the optimal set and exploiting the cardinality of each of these choice sets (§ 4.3). We conclude by a synthesis of these comparisons (§ 4.4). Transitive closures are next introduced (§ 5.1), and equivalence results are proved, inducing linear algorithms for computing choice sets based on relative choice axioms for any kind of digraphs (§ 5.2). We close section 5 with a practical application in the postal service (§ 5.3). Finally (§ 6), we conclude the paper based on relative choice axioms for any kind of digraphs (§ 5.2). We close section 5 with a synthesis of our results and some further remarks and perspectives. To balance the lengths of the sections and improve readability of the paper, some proofs are provided as supplementary material in the Appendix.

2. Preliminaries

2.1. Crisp Binary Preference Relations, Digraphs and Choice Sets

Throughout this paper, \( S \) denotes the non-empty finite set of all alternatives. A **crisp binary preference relation (CBPR)** \( \succcurlyeq \) of an individual on \( S \) is a reflexive binary relation on \( S (\iff \succcurlyeq \subseteq S \times S \ )\) and \( \forall x \in S, (x, x) \in \succcurlyeq \)\), translating preferential judgements of the individual between pairs of alternatives of \( S \). We note that \( x \succcurlyeq y \) instead of \( (x, y) \in \succcurlyeq \), and \( \text{not}(x \succcurlyeq y) \) to designate \( (x, y) \notin \succcurlyeq \). For every pair of elements \( x \) and \( y \) of \( S \), the assertion “\( x \succcurlyeq y \)” means “\( x \) is at least as good as \( y \) for the considered individual.” A CBPR \( \succcurlyeq \) determines a partition of \( S \times S \) into four fundamental relations:

- **(indifference)** \( x \equiv y \iff (x \succcurlyeq y \text{ and } y \succcurlyeq x) \) for all \( x, y \in S \).
- **(strict preference)** \( x \succ y \iff (x \succcurlyeq y \text{ and } \not(y \succcurlyeq x)) \) for all \( x, y \in S \).
- **(strict aversion)** \( x \ll y \iff y \succ x \) for every \( x, y \in S \).
- **(incomparability)** \( x \nmid y \iff (\not(x \succcurlyeq y) \text{ and } \not(y \succcurlyeq x)) \) for all \( x, y \in S \).

For every non-empty \( A \subseteq S \), the **restriction** of \( \succcurlyeq \) to \( A \) is the preference relation \( \succcurlyeq_{|A} \) defined as follows: \( \succcurlyeq_{|A} = \{(x, y) \in A \times A \text{, such that: } x \succcurlyeq y \} \). We do not specify the restriction, but instead the context enables us to identify the targeted subset of \( S \). A preference relation \( \succcurlyeq \) is:

- **P-acyclic** iff \( \forall t \geq 2 \text{ and } \forall x_1, x_2, \ldots, x_t \in S, \text{ we have } x_1 \succcurlyeq x_2 \succcurlyeq \ldots \succcurlyeq x_t \iff \not(x_t \succcurlyeq x_1) \)
- **acyclic** iff it is P-acyclic and antisymmetric\(^3\) = it includes no circuit of length \( \geq 2 \).
- **an equivalence relation** iff it is reflexive, symmetric and transitive.

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\(^3\) We note that a binary relation \( \succcurlyeq \) is **reflexive** iff \( x \succcurlyeq x \), for all \( x \in S \); **symmetric** iff \( x \succcurlyeq y \iff y \succcurlyeq x \), for all \( x, y \in S \); **antisymmetric** iff \( x \succcurlyeq y \iff \not(y \succcurlyeq x) \), for all \( x, y \in S \) such that \( x \neq y \); **asymmetric** iff \( x \succcurlyeq y \iff \not(y \succcurlyeq x) \), for all \( x, y \in S \); **transitive** iff \( x \succcurlyeq y \) and \( y \succcurlyeq z \implies x \succcurlyeq z \), for all \( x, y, z \in S \); and **complete** (or **total**) iff \( x \succcurlyeq y \) or \( y \succcurlyeq x \), for all \( x, y \in S \) and \( x \neq y \).
Similarly, we define sets satisfying a property maximality $S$.

The maximal (w.r.t. inclusion) set of ($\bowtie$)

- a partial preorder (or partial weak order, or quasi ordering) iff it is reflexive and transitive.
- a complete preorder (or weak order, or ordering) iff it is reflexive, transitive and complete.
- a complete order iff it is reflexive, antisymmetric, transitive and complete.
- a tournament iff it is reflexive, antisymmetric and complete.
- a weak tournament iff it is reflexive and complete.

The couple $(S, \bowtie)$ formally corresponds with the concept of simple reflexive directed graph, referred to below as digraph. In this article, we will use the following definitions from set theory (Bang-Jensen and Gutin, 2001): a set $U$ satisfying a property $P$ is maximum with respect to (w.r.t.) cardinality (resp. maximal w.r.t. inclusion) according to property $P$ if there is no set $U'$ satisfying $P$ and $|U'| > |U|$ (resp. $U \subset U'$).

Similarly, we define sets satisfying a property $P$ and minimum w.r.t. cardinality (resp. minimal w.r.t. inclusion). We use the notations $\text{MXM}(P)$, $\text{MXL}(P)$, $\text{MNM}(P)$ and $\text{MNL}(P)$ to point out such respective sets. However, we will use $\text{UNION}(P)$ (resp. $\text{INTER}(P)$) to point out the union (resp. intersection) of subsets satisfying property $P$.

We denote $R_{\text{CBPR}, S}$ the set of preference relations on $S$. A choice set $C(S, \bowtie)$ on $S$ according to $\bowtie$ is a non-empty subset of $S$, interpreted here as the set of best alternatives for $(S, \bowtie)$. It is usual to look for choice sets that fulfil some properties, called choice axioms.

In the same way, non-empty intersection may be defined. In the particular case of axioms $CA_1$ and $CA_2$ characterising each only one choice set, resp. $C_1(S, \bowtie)$ and $C_2(S, \bowtie)$, we have: $CA_1$ is said to be equivalent to $CA_2$ if $\forall \bowtie \in R_{\text{CBPR}, S}$, $C_1(S, \bowtie) = C_2(S, \bowtie)$. In this case, we use the term “choice axiom” to designate directly the characterized choice set.

In the following, we introduce the choice sets and axioms that are universally accepted.

### 2.2. Usual Choice Axioms

The axioms commonly used in the literature are optimality, maximality, strong optimality and strong maximality. We denote these by usual choice axioms with the following formal definition:

A subset $A$ of $S$ has the optimality (resp. maximality, strong optimality and strong maximality) property according to the preference relation $\bowtie$ iff:

$$\forall x \in A \text{ and } \forall y \in S \setminus \{x\}, \ x \bowtie y . \ (\text{optimality} = 0) \quad (1)$$

$$\forall x \in A \text{ and } \forall y \in S \setminus \{x\}, \ \text{not}(y > x) . \ (\text{maximality} = M) \quad (2)$$

$$\forall x \in A \text{ and } \forall y \in S \setminus \{x\}, \ \text{not}(y \bowtie x) . \ (\text{strong maximality} = SM) \quad (3)$$

$$\forall x \in A \text{ and } \forall y \in S \setminus \{x\}, \ x > y . \ (\text{strong optimality} = SO) \quad (4)$$

The maximal (w.r.t. inclusion) set of $(S, \bowtie)$ according to the axiom $CA_4$ with $CA \in \{0, M, SO, SM\}$, is called the $\text{MXL}(CA)$-set. Therefore, the optimal set $B(S, \bowtie)$ is then called the $\text{MXL}(0)$-set, and the maximal set (or efficient-set (Roy and Bouyssou, 1993; White, 1977)) $M(S, \bowtie)$ is then called the $\text{MXL}(M)$-set. These subsets have the properties summarised below.

### Proposition 1

- **Uniqueness:** $\forall CA \in \{0, M, SO, SM\}$, the $\text{MXL}(CA)$-set is unique if it exists.
• **Existence:**
  a) There exist digraphs with empty $\text{MXL}(CA)$-sets, $\forall CA \in \{O, M, SO, SM\}$; 
  b) $\text{MXL}(SO)$-set $\neq \emptyset \iff \text{MXL}(SO)$-set is a singleton.

• **Internal structure:** The optimal (resp. strong maximal, resp. maximal) elements are pairwise indifferent (resp. incomparable, resp. indifferent or incomparable).

• **Cross comparison:**
  c) $\text{MXL}(SO)$-set $\neq \emptyset \iff [\text{MXL}(O)$-set $\neq \emptyset \text{ and } \text{MXL}(SM)$-set $\neq \emptyset] \Rightarrow \text{MXL}(SO)$-set $= \text{MXL}(CA)$-set, $\forall CA \in \{O, M, SM\}$.
  d) $\text{MXL}(SM)$-set $\cup \text{MXL}(O)$-set $\subseteq \text{MXL}(M)$-set.
  e) Given $\text{MXL}(O)$-set $\neq \emptyset$. Then, $\text{MXL}(M)$-set $\neq \emptyset$ and moreover, 
     $[\forall (x, y) \in M(S, \succ) \times S, \not(x \parallel y)] \iff \text{MXL}(O)$-set $= \text{MXL}(M)$-set.
  f) Given $\text{MXL}(SM)$-set $\neq \emptyset$. Then, $\text{MXL}(M)$-set $\neq \emptyset$ and moreover, 
     $[\forall (x, y) \in M(S, \succ) \times S \setminus \{x\}, \not(x \approx y)] \iff \text{MXL}(SM)$-set $= \text{MXL}(M)$-set.

This proposition is straightforward to prove. Some of its properties are well known in the literature as the internal structure of $\text{MXL}(CA)$-sets, or the uniqueness of the strong optimal element in tournaments which was exploited to define voting winners (Condorcet, Kemeny, etc.), or else the equality conditions between the $\text{MXL}(CA)$-sets (see Sen, 1970, 1997, 2002 chap. 4).

Fig. 1. **Examples of digraphs serving as support for usual choice axioms.**

This figure illustrates these choice sets: So, the digraph (a) verifies every $\text{MXL}(CA)$-set is empty. The digraph (b) entails that the optimal set $\{y\}$ is smaller than the maximal set $S$, while both of the other possible choice sets are empty. The digraph (c) entails that the strong maximal set $\{z\}$ is smaller than the maximal set $S$, while both of the other possible choice sets are empty. The digraph (d) entails that only the maximal set $\{w, x, z\}$ is non-empty.

### 3. Relative Choice Axioms

How can we choose the best quality elements from a couple $(S, \succ)$, when the usual choice sets are empty? The main part of the MCDA literature deals with either choice sets satisfying the von Neumann-Morgenstern domination (see survey of Ghoshal *et al.*, 1998) or with maximal set of classical transitive closure (Sen, 1986). This section is devoted to direct extensions of the top-cycle from tournaments to digraphs. We define and characterise them.

#### 3.1. Definitions and Basic Properties

A **non-empty** subset $A$ of $S$ is a **relatively optimal set** (resp. a **relatively maximal set**), and noted $\text{RO}$-set (resp. $\text{RM}$-set) according to a binary relation of preference $\succ$ if and only if:

\[
\forall x \in A \text{ and } \forall y \in S \setminus A, \quad x \succ y. \quad (\text{relative optimality} = \text{RO}) \tag{5}
\]

\[
\forall x \in A \text{ and } \forall y \in S \setminus A, \quad \not(y > x). \quad (\text{relative maximality} = \text{RM}) \tag{6}
\]

In the same way, we obtain the following relative choice axioms associated with strong maximality and strong optimality: 

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4 Note that loops (reflexive arcs) are omitted from the figure.
∀x ∈ A and ∀y ∈ S \ A, not(y ≽ x). (relative strong maximality = RSM) (7)
∀x ∈ A and ∀y ∈ S \ A, x ≻ y. (relative strong optimality = RSO) (8)

These axioms weaken the usual choice axioms (see § 4.2 for further discussion). In terms of the mathematical formulation of the definitions, relative choice axioms are distinguished from usual ones only by the definition set of y. The former is the set S \ A, and the latter is the set S \ \{x\}. This small variation induces a large difference between characterised subsets.

The table in Fig. 2 summarises properties of the binary relation inducing trivial equivalences (coincidences) between the four different choice axioms RO, RM, RSO and RSM:

<table>
<thead>
<tr>
<th></th>
<th>RO</th>
<th>RM</th>
<th>RSO</th>
<th>RSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSM</td>
<td>tournament</td>
<td>antisymmetric relation</td>
<td>complete relation</td>
<td></td>
</tr>
<tr>
<td>RSO</td>
<td>antisymmetric relation</td>
<td>tournament</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RM</td>
<td>complete relation</td>
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<td></td>
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</tr>
</tbody>
</table>

Fig. 2. Digraph properties involving coincidences between relative choice axioms.

Some basic logical properties linking the relative sets RO, RM, RSO and RSM are given below.

**Proposition 2.** Every RO-set is an RM-set, every RSO-set is both an RO-set and an RSM-set, and every RSM-set is an RM-set. Let CA be an axiom belonging to \{RO, RM, RSO, RSM\}. Then there always exists at least one CA-set, otherwise S is empty because S is itself a CA-set; And, every union and non-empty intersection of CA-sets is a CA-set (i.e. the set of all the CA-sets, plus the empty set, structured by inclusion, is a lattice).

The proof is immediate. Fig. 3 represents, via the order (Hasse) diagrams (Trotter, 1992), the lattices induced by relative choice axioms from digraph (b) of Fig. 4.

Fig. 3. Lattices induced by the relative choice sets (after adding the empty set) from digraph (b) of Fig. 4.

Note that the lower the level which an alternative first appears in a Hasse diagram, then the more likely the alternative is among the best solutions. So, in the previous example, the alternatives x, y, z appear at the lowest level (just after the empty set) of the four diagrams, while w always appears at their highest level. Indeed, w is dominated by all other alternatives (Fig. 4(b)). Such rankings are good tools for supporting the next stage of robustness analysis (Roy, 1998, 2010; Roy and Bouysou, 1993). Another particularity of these Hasse diagrams is that their restriction to vertices where a given alternative x appears is structured as a lattice, and its lower bound is the smallest relative set containing x. For example, the lower bound of the sub-lattice of Fig. 3(b) containing x is \{x, y\}.

Now, we point out minimal and minimum relative sets defined below.
A relatively optimal set $MRO$ is minimal with respect to inclusion, denoted as $MNL(RO)$-set, if there exists no relatively optimal set strictly included in $MRO$. A relatively optimal set $MRO$ is minimum with respect to cardinality, denoted as $MNN(RO)$-set, if $|MRO|$ is minimal. In the same meaning, we also define the $MNL(CA)$-set and the $MNM(CA)$-set, with $CA \in \{RM, RSO, RSM\}$.

**Proposition 3.** The axiom $MNL(RSO)$ identifies a unique non-empty subset, contrary to the other minimal relative choice axioms $MNL(CA)$, with $CA \in \{RO, RM, RSM\}$.

**Proof:** Concerning the non-uniqueness of $MNL(RO)$, $MNL(RM)$ and $MNL(RSM)$, digraph (a) of Fig. 4 identifies two $MNL(RO)$-sets $\{x_1, x_3\}$ and $\{x_2, x_4\}$ and four $MNL(RM)$-sets $\{x_1, x_2\}$, $\{x_3\}$ and $\{x_4\}$. Digraph (b) of Fig. 4 identifies two $MNL(RSM)$-sets $\{x, y\}$ and $\{z\}$. We now show the uniqueness of the $MNL(RSO)$-set. Assume there exists a digraph $(S, \succ)$ such that $MNL(RSO)$ identifies at least two non-empty subsets $MRSO_1$ and $MRSO_2$ verifying $MRSO_1 \setminus MRSO_2 \neq \emptyset$. Then, $MRSO_1$ and $MRSO_2$ are disjoint (Proposition 2). Let $x \in MRSO_1 \subseteq S \setminus MRSO_2$. By definition, $\forall y \in MRSO_2 \subseteq S \setminus MRSO_1$, $x \succ y$. And, as $x \notin MRSO_2$, then $y \succ x$. A contradiction. Hence, $MNL(RSO)$ identifies a unique non-empty set when $|S| \geq 1$. \hfill \Box

Note that cardinality is not an invariant among the $MNL(RO)$-sets of a couple $(S, \succ)$. Digraph (c) shows a couple $(S, \succ)$ with two $MNL(RO)$-sets $\{x_5\}$ and $\{x_1, \ldots, x_3\}$. The same remarks apply to $MNL(RM)$-sets and $MNL(RSM)$-sets.

According to Proposition 3, $MNL(RSO)$ identifies a unique subset, and can thus be used as choice axiom. For the other minimal relative sets associated with $RO, RM$ and $RSM$, so as to define axioms identifying a unique subset, we will consider two unions of $MNL(CA)$, with $CA \in \{RO, RM, RSM\}$: the union of all minimum (i.e. smallest) relative sets, or union of $MNL(CA)$-sets), and the union of all minimal relative sets, or union of $MNL(CA)$). The uniqueness of these unions is guaranteed by their very definition.

**Proposition 4.** The axiom $MNL(CA)$ is idempotent, with $CA \in \{RO, RM, RSO, RSM\}$: Given a structured finite set $(S, \succ)$, we denote by $MICAS(U, \succ)$ any $MNL(CA)$-set of the non-empty subset $U$ of $S$. Then, we have: $MICAS(MICAS(S, \succ), \succ) = MICAS(S, \succ)$.

**Proof:** We prove this property for $MNL(RSO)$. We note $MIRSOS(S, \succ)$ the $MNL(RSO)$-set of $(S, \succ)$. Suppose $MIRSOS(S, \succ) \setminus MIRSOS(MIRSOS(S, \succ), \succ) \neq \emptyset$. Then, $\forall x \in MIRSOS(MIRSOS(S, \succ), \succ)$ and $\forall y \in MIRSOS(S, \succ) \setminus MIRSOS(MIRSOS(S, \succ), \succ)$, $x \succ y$. Moreover, since $x$ and $y$ belong to $MIRSOS(S, \succ)$, then $\forall z \in S \setminus MIRSOS(S, \succ)$, $x \succ z$ and $y \succ z$. Finally, $\forall x \in MIRSOS(MIRSOS(S, \succ), \succ)$ and $\forall z \in S \setminus MIRSOS(MIRSOS(S, \succ), \succ)$, $x \succ z$. Accordingly, $MIRSOS(MIRSOS(S, \succ), \succ)$ is an $RSO$-set for $(S, \succ)$ strictly included in $MIRSOS(S, \succ)$ a $MNL(RSO)$-set in $(S, \succ)$. This contradicts the minimality of $MIRSOS(S, \succ)$ in $(S, \succ)$. Therefore, $MNL(RSO)$ is idempotent. This property can be proved for the other axioms by simply adapting the above proof. \hfill \Box

The idempotence of $\text{union}(MNL(CA))$ and $\text{union}(MNN(CA))$ can be deduced directly from Proposition 4. Idempotence is a kind of ‘procedural minimality’: The recursive use of idempotent axioms, by isolation of chosen elements, does not improve the choice (i.e.
does not make the choice set smaller). This property gives a kind of stability\(^5\) to the identified choice sets.

### 3.2. Literature Review

Most articles in economics focus on tournaments and complete relations.

In the case of tournaments, the four axioms RO, RM, RSO and RSM coincide and the MNL(RSO)-set is called the **top-cycle set** (Miller, 1977).

In the case of complete relations, RM and RO coincide and the \(\text{UNION}(\text{MNL}(\text{RM}))\)-set is called either the **GOCHA**\(^6\)-set, the **union of minimal undominated sets**, the **Schwartz-set**, or the **strong top cycle set** (Schwartz, 1972, 1986; Deb, 1977; Peris and Subiza, 1999; Duggan, 2007). The RSM and RSO axioms coincide and are called the **Condorcet transitivity axiom** (Peris and Subiza, 1999). The \(\text{MNL}(\text{RSO})\)-set is called either the **GETCHA**\(^7\)-set, the **minimal P-dominant subset**, the **Smith-set**, or the **weak top cycle set** (Schwartz, 1972; Subiza and Peris, 2000; Duggan, 2007). Subiza and Peris (2005a) show a strong link between the optimal set and the \(\text{UNION}(\text{MNM}(\text{RM}))\)-set, also called **rational top-cycle**, that we generalize to digraphs in § 4.2. Deb (1977), Schwartz (1986) characterise the GOCHA-set and the GETCHA-set as the maximal elements of two different transitive closures of the initial complete CBPR. We generalise these results to digraphs in § 5.1.

In the case of digraphs, Brandt et al. (2009) investigate the relationship between the \(\text{MNL}(\text{RSO})\)-set and the \(\text{UNION}(\text{MNL}(\text{RM}))\)-set. They also characterise their low computational complexity. We extend these complexity results to the entire family of relative choice axioms in § 5.2.

To understand their relevance, in the rest of this section, we look at the internal properties of minimal relative choice sets (§ 3.3), and we make an accurate set-theoretical comparison in § 4.

### 3.3. Characterisation of the Internal Structure of Minimal Relative Choice Sets

We focus on the internal structure of \(\text{MNL}(\text{CA})\)-sets, i.e. the properties of sub-digraphs of \((S, \succ)\) induced by the \(\text{MNL}(\text{CA})\)-sets, for every \(\text{CA} \in \{\text{RO}, \text{RM}, \text{RSO}, \text{RSM}\}\). We first simplify the different sentences and proofs, by defining the concept of attitude (Belmandt, 1993) and its link with our targeted choice axioms.

We denote \(\mathcal{FA} = \{\text{indifferent}, \text{better}, \text{worse}, \text{incomparable}\}\) as the set of **fundamental attitudes** of CBPRs. The set of proper subsets of \(\mathcal{FA}\) (i.e. subsets different from the empty set and \(\mathcal{FA}\)) is denoted by \(\mathcal{PR}(\mathcal{FA})\), and its elements are called the **attitudes** of the CBPRs. We use also the word **outrank** when referring to the attitude \{better, indifferent\}. We will write \(x \alpha(\succ) y\), with \(\alpha(\succ)\) identifying the attitude \(\alpha\) associated with the CBPR \(\succ\). Therefore, better(\(\succ)\) is equal to \(\succ\), and so on.

---

5 Stability here is a general principle used in rational choice theory to characterise choice functions. Do not confuse this with the internal stability (see § 5.3; von Neumann and Morgenstern, 1944; Berge, 1973; Ghoshal et al., 1998). Informally, this principle generalises the idempotence to any non-empty subset of the alternative set \(S\) (Suzumura, 1983).

6 GOCHA stands for Generalized Optimal Choice Axiom.

7 GETCHA stands for Generalized Top Choice Axiom.
Definition 1. We associate an attitude $\alpha(\text{CA})$ of $\mathcal{P}(\mathcal{A})$ to each choice axiom $\text{CA}$, with $\text{CA} \in \{\text{RO}, \text{RM}, \text{RSM}, \text{RSM}\}$, as follow: $\alpha(\text{RO}) = \{\text{better, incomparable}\}$, $\alpha(\text{RM}) = \{\text{better}\}$, $\alpha(\text{RSM}) = \{\text{better}\}$.

This definition will be also used in § 5.1 to explain the strong link between $\text{MNL}($ $\text{CA}$ $)$-sets and some transitive closures. Proposition 3 says that there usually exists only one $\text{MNL}(\text{RSM})$-set, in opposition with the other minimal relative sets. The following proposition gives some details about the links between two minimal $\text{CA}$-sets, for a fixed $\text{CA} \in \{\text{RO}, \text{RM}, \text{RSM}\}$.

Proposition 5. Given a choice axiom $\text{CA} \in \{\text{RO}, \text{RM}, \text{RSM}\}$ and two different $\text{MNL}($ $\text{CA}$ $)$-sets $\text{MICAS}_1$ and $\text{MICAS}_2$ of a digraph $(S, \succ)$, with $S$ finite and non-empty, then:

(a) Their intersection is empty: $\text{MICAS}_1 \cap \text{MICAS}_2 = \emptyset$.

(b) Two alternatives which do not belong to the same $\text{MNL}(\text{CA})$-set satisfy the following: $\forall (x, y) \in \text{MICAS}_1 \times \text{MICAS}_2$, $x \alpha_1(\succ) y$, with $\alpha_1 \in \alpha(\text{RSM}) \setminus \alpha(\text{CA})$.

In other words, assertion (b) means that two elements of two different minimal RO-sets (resp. RSM-sets, resp. RM-sets) are indifferent (resp. incomparable, resp. indifferent or incomparable).

Proof: Assertion (a), we reason by contradiction. First, we know that $\text{MICAS}_1 \setminus \text{MICAS}_2 \neq \emptyset$ and $\text{MICAS}_2 \setminus \text{MICAS}_1 \neq \emptyset$ (i.e. none can be included in the other), because $\text{MICAS}_1$ and $\text{MICAS}_2$ are different and minimal w.r.t. inclusion. Consequently, they cannot contain a $\text{CA}$-set. Second, according to Proposition 2, $\text{MICAS}_1 \cap \text{MICAS}_2$ is a $\text{CA}$-set. Therefore, $\text{MICAS}_1$ and $\text{MICAS}_2$ strictly contain a $\text{CA}$-set. This contradicts the initial assumption, hence $\text{MICAS}_1 \cap \text{MICAS}_2 = \emptyset$.

Assertion (b), we only consider the case $\text{CA} = \text{RO}$; the proof is analogous for choice axioms RM and RSM. To prove that $x \simeq y \ \forall (x, y) \in MRO_1 \times MRO_2$, we show first that $x \succ y$, and next $y \succ x$. We have $x \succ y$ by definition of the RO-set $MRO_1$: $x \in MRO_1$ and $y \notin MRO_2 \setminus MRO_1 \subseteq S \setminus MRO_1$. This is likewise for $y \succ x$ by definition of the RO-set $MRO_2$: $y \in MRO_2$ and $x \notin MRO_1 \setminus MRO_2 \subseteq S \setminus MRO_2$. Finally, $x$ and $y$ are indifferent for any $(x, y) \in MRO_1 \times MRO_2$.

We now characterise the $\text{MNL}(\text{CA})$-sets.

Proposition 6. $\forall \text{CA} \in \{\text{RO}, \text{RM}, \text{RSM}, \text{RSM}\}$, suppose we have a $\text{CA}$-set $\text{CAS}$ of a digraph $(S, \succ)$ with $S$ finite and non-empty, then $\text{CAS}$ is a $\text{MNL}(\text{CA})$-set of $(S, \succ)$ iff:

(a) $\text{CAS}$ is reduced to one element; or

(b) $\text{CAS}$ is made up of two elements $x$ and $y$ satisfying: $x \alpha_1(\succ) y$, with $\alpha_1 \in \alpha(\text{CA}) \setminus \{\text{better}\}$; or else

(c) The cardinality of $\text{CAS}$ is greater than 2 and the digraph $(\text{CAS}, \alpha(\text{CA})(\succ))$ is strongly connected.

The proof is available in Section A of the Appendix (supplementary material).

Proposition 7. If $\text{CAS}$ is a $\text{MNL}(\text{RM})$-set, then its cardinality is always different from 2.

This proposition is a direct consequence of the propositions 5 (b) and 6 (b).

4. Set-Theoretical Comparisons of Choice Axioms

In this section, we carry out a set-theoretical comparison of minimal relative choice axioms. Next, we provide arguments to confirm the interest of relative choice axioms in comparison with the usual axioms. After, we compare the minimum relative choice
axioms, and we detail a combination of them as a pertinent candidate for choice. At last, we make a synthesis of these results and advocate some choice axioms.

4.1. Cross Comparison of Minimal Relative Choice Axioms

In this section, we make a cross comparison of MNL(CA)-sets, with CA ∈ {RO, RM, RSO, RSM}. The following proposition describes the first inclusions.

**Proposition 8.** Given a digraph (S, ≽) such that S is finite and non-empty, then:

(a) Every MNL(CA)-set contains a MNL(RM)-set, with CA ∈ {RO, RSM}.
(b) UNION(MNL(RM))-set ∩ UNION(MNL(CA))-set ≠ ∅, with CA ∈ {RO, RSM}.
(c) Every MNL(CA)-set is included in the MNL(RSO)-set, with CA ∈ {RO, RM, RSM}.

**Proof:** These assertions follow from Proposition 2. For assertion (a), every RO-set and RSM-set – and particularly every MNL(RO)-set and MNL(RSM)-set – is an RM-set, and every RM-set contains a MNL(RM)-set. Assertion (b): This is a direct corollary of assertion (a). For assertion (c), we suppose that there exists a MNL(CA)-set MCA not contained in the MNL(RSO)-set. They are disjoint because the MNL(RSO)-set is a CA-set (Proposition 2) and the intersection of two CA-sets – MCA and MNL(RSO)-set – is a CA-set, contradicting the minimality w.r.t. inclusion of MCA. Moreover, MCA ∩ MNL(RSO)-set = ∅ ⇔ ∀ x ∈ MCA and ∀ y ∈ MNL(RSO)-set, x ≽ y and x ≽ y for CA = RO (resp. not(y ≽ x) for CA = RM, resp. not(y ≽ x) for CA = RSM). They all lead to contradictions. Finally, any MNL(CA)-set is included in the MNL(RSO)-set.

Unfortunately, the converse inclusions are not always true. There exists couples (S, ≽) with MNL(RM)-sets not contained in a MNL(RO)-set, for CA ∈ {RO, RSM}. Illustrating examples are given in Fig. 5. Digraph (a) has two MNL(RM)-sets {x} and {y} and one MNL(RO)-set {x}, implying the MNL(RM)-set {y} is not contained in a MNL(RO)-set. Digraph (b) has two MNL(RM)-sets {y} and {z} and one MNL(RSM)-set {z}, implying that the MNL(RM)-set {y} is not contained in a MNL(RSM)-set. The following theorem generalizes these examples.

**Theorem 1.** Given a digraph (S, ≽) with S finite and non-empty and a couple of choice axioms (CA1, CA2) ∈ {(RO, RSM), (RS, RO)}, then we have the following (see Fig. 6):

(a) The uniqueness of the MNL(CA2)-set is induced by the multiplicity of the MNL(CA1)-sets:

![Fig. 5. Comparison of MNL(RO)-sets and MNL(RSM)-sets.](image1)

![Fig. 6. Synthesis sketch of both kinds of configurations for minimal relative sets in digraphs.](image2)
The number of $\text{MNL}(\text{CA}_1)$-sets is $> 1 \Rightarrow$ There exists one and only one $\text{MNL}(\text{CA}_2)$-set.

(b) The number of $\text{MNL}(\text{CA}_1)$-sets is $> 1 \Rightarrow \text{UNION}(\text{MNL}(\text{CA}_1))$-set $\subseteq \text{MNL}(\text{CA}_2)$-set.

(c) The number of $\text{MNL}(\text{CA}_1)$-sets is $> 1 \Rightarrow \text{UNION}(\text{MNL}(\text{RM}))$-set $\subseteq \text{MNL}(\text{CA}_2)$-set.

(d) The uniqueness of $\text{MNL}(\text{RO})$-set and $\text{MNL}(\text{RSM})$-set $\Rightarrow$ Either $\text{MNL}(\text{RO})$-set $\subseteq \text{MNL}(\text{RSM})$-set or the converse is true.

The proof is presented in Section B of the Appendix (supplementary material).

We remark that the intersection of a $\text{MNL}(\text{RO})$-set $\text{MRO}$ and a $\text{MNL}(\text{RSM})$-set $\text{MRSM}$ can contain several $\text{MNL}(\text{RM})$-sets, as illustrated in digraph (a) of Fig. 7, where $\text{MRO} = \text{MRSM} = S$.

The above results lead to several direct consequences concerning the number of minimal relative sets and the size of the relative choice sets. In particular, the number of $\text{CA}_1$-sets and $\text{MNL}(\text{CA}_1)$-sets are at least equal to the number of $\text{CA}_2$-sets and $\text{MNL}(\text{CA}_2)$-sets, with $(\text{CA}_1, \text{CA}_2) \in \{(\text{RM}, \text{RSM}), (\text{RM}, \text{RO}), (\text{RSM}, \text{RSM}), (\text{RO}, \text{RSM})\}$. Moreover, the size of the $\text{UNION}(\text{MNL}(\text{CA}_1))$-set is smaller than the size of the $\text{UNION}(\text{MNL}(\text{CA}_2))$-set. Finally, the number of $\text{MNL}(\text{CA})$-sets, with $\text{CA} \in \{\text{RO}, \text{RM}, \text{RSM}, \text{RSM}\}$, is upper bounded by the size of $S$.

![Fig. 7.](image)

4.2. Comparing Relative and Usual Choice Axioms

Minimal relative choice sets have an outside behaviour* corresponding to usual choice sets. However, contrary to the latter, minimal relative choice sets allow for contradictory (conflicting) inside elements. In this sense, using relative axioms is a prudent choice.

The following theorem sketches the logical link between usual choice axioms and homologue relative choice axioms. It generalizes the theorem 1 of Subiza and Peris (2005a) to not necessarily complete digraphs.

**Theorem 2.** Given a binary preference relation on a non-empty set $(S, \succ)$, and a choice axiom $\text{CA} \in \{O, M, SO, SM\}$, we have $\text{MNL}(\text{CA})$-set $\neq \emptyset \Rightarrow \text{UNION}(\text{MNM}(\text{RCA}))$-set $= \text{MNL}(\text{CA})$-set.

**Proof:** If $\text{MNL}(\text{CA})$-set is non-empty, then each element of the $\text{MNL}(\text{CA})$-set defines a singleton $\text{MNL}(\text{RCA})$-set, which is in fact a $\text{MNM}(\text{RCA})$-set since its cardinal is 1.

Example (c) of Fig. 4 has 2 $\text{MNL}(\text{RO})$-sets, one of them being a singleton and defining the optimal set corresponding to the only $\text{MNM}(\text{RO})$-set. Note, however, that $\forall \text{CA} \in \{O, M, SO, SM\}$, if there exists no $\text{MNL}(\text{CA})$-set of cardinality 1, then the $\text{UNION}(\text{MNM}(\text{RCA}))$-set can be used as a choice set, and it exists for every non-empty set $S$ structured by any CBPR $\succ$. Moreover, this axiom is idempotent (see § 3.1) and has an interesting internal structure (§ 3.3). By considering only $\text{MNM}(\text{RCA})$-sets instead of $\text{MNL}(\text{RCA})$-sets, we reduce the size of the choice set, hence the hesitation of the decision-maker.

* That is, the preferential behaviour of elements inside the characterised subsets, towards elements outside these subsets.
We now consider the question of whether there exists a stronger logical link between these two families of axioms. The following proposition provides the first answer.

Proposition 9. Given a digraph \((S, \succeq)\) such that \(0 < |S| < +\infty\), and a couple of choice axioms \((CA_1, CA_2) \in \{(O, SM), (SM, O)\}\), then if the \(\text{MXL}(CA_1)\)-set is non-empty, then there exists only one \(\text{MNL}(RC_{A_2})\)-set. Moreover, we have the following inclusion: \(\text{MXL}(CA_1)\)-set \(\subseteq\) \(\text{MNL}(RC_{A_2})\)-set.

Proof: This result stems directly from Theorem 2 and Theorem 1 (a) and (b).

4.3. From Cross Comparison to Combination of Minimum Relative Choice Axioms

We now compare the \(\text{UNION}(\text{MNM}(CA))\) axioms, with \(CA \in \{RO, RM, RSM\}\).

We remark that if an optimal set exists, then from Proposition 1 (d) and Theorem 2: \(\text{UNION}(\text{MNM}(RO))\)-set \(\subseteq\) \(\text{UNION}(\text{MNM}(RM))\)-set. This observation complements Proposition 8 (b) on minimal relative sets. However, the reader might ask whether this link can be generalised when the optimal set is empty. Here is the answer:

Proposition 10. Given a choice axiom \(CA \in \{RO, RSM\}\), then, there exist digraphs \((S, \succeq)\) such that:

(a) \(\text{UNION}(\text{MNM}(RM))\)-set \(\cap\) \(\text{UNION}(\text{MNM}(CA))\)-set = \(\emptyset\).

(b) \(\text{UNION}(\text{MNM}(RM))\)-set \(\cap\) \(\text{UNION}(\text{MNL}(CA))\)-set = \(\emptyset\).

Proof: The first non-overlapping is true if the second is. We then only show the assertion (b). For \(CA = RO\), Fig. 7 (b) provides an example where the \(\text{UNION}(\text{MNL}(RO))\)-set \{x_1, x_2, x_3\} is disjoint with the maximal set \{y\}, with \(S = \{x_1, x_2, x_3, y, z\}\) and a CBPR \(\succeq\) verifying: \(x_i \sim y\) and \(x_i \succ z\) for all \(i \in \{1, 2, 3\}\), \(x_1 \succ x_2 \succ x_3 \succ x_1\) and the other non-reflexive couple of alternatives are incomparable. An equivalent example is available for \(CA = RSM\), from Fig. 7 (b), by replacing indifference by incomparability and vice versa.

During the decision process, the decision-maker is led to compare the chosen solutions in order to identify the most ‘satisficing’ solutions (Simon, 1977). Accordingly, the less he has to compare, the better off he will be. Therefore, Theorem 2 and Proposition 10 suggest that he should use an alternative choice set, based on a combination of relative maximality, relative optimality and relative strong maximality. Consider the following examples. In the digraphs of Fig. 8, the most appropriate choice set is \{x_5, x_6, x_7\}. This choice set is the smaller set between the \(\text{UNION}(\text{MNM}(RM))\)-set and the \(\text{UNION}(\text{MNM}(RO))\)-set (for first digraph of Fig. 8) and the \(\text{UNION}(\text{MNM}(RSM))\)-set (for second digraph of Fig. 8). Therefore, we consider the following choice axiom:

\(\text{MNM}(\text{UNION}(\text{MNM}(RM)), \text{UNION}(\text{MNM}(RSM)), \text{UNION}(\text{MNM}(RO)))\).

The characterised choice set is obtained by choosing the minimum between the \(\text{UNION}(\text{MNM}(CA))\)-sets with \(CA \in \{RO, RSM, RM\}\).

![Fig. 8. Digraphs illustrating disjunctions between minimum relative choice sets and the maximal set. 4](image-url)
The following results detail the properties of this axiom. First of all, we show that for every digraph, either the \( \text{UNION}(\text{MNM}(\text{RSM})) \)-set is included in the \( \text{UNION}(\text{MNM}(\text{RO})) \)-set or the converse is true. In fact, the next proposition generalises these latter comparisons, on more relative choice sets, and will be useful for synthesis in § 4.4. Afterward, we give some other main properties of the \( \text{MNM}(\text{UNION}(\text{MNM}(\text{CA}))) \), with \( CA \in \{\text{RO}, \text{RSM}, \text{RM}\} \) axiom.

**Proposition 11.** Given a digraph \((S, \succ)\) such that \(0 < |S| < +\infty\), then at least one of the following assertions is true:

(a) \( \text{UNION}(\text{MNL}(\text{RSM}))-\text{set} \subseteq \text{UNION}(\text{MNM}(\text{RO}))-\text{set} \),

(b) \( \text{UNION}(\text{MNL}(\text{RO}))-\text{set} \subseteq \text{UNION}(\text{MNM}(\text{RSM}))-\text{set} \).

**Proof:** This proposition is a direct consequence of Theorem 1. We decompose the demonstration according to the multiplicity of the \( \text{MNL}(\text{CA}) \)-sets, with \( CA \in \{\text{RO}, \text{RSM}\} \). For every digraph, we have three possibilities (we cannot have at the same time several \( \text{MNL}(\text{RO}) \)-sets and several \( \text{MNL}(\text{RSM}) \)-sets; see Theorem 1 (a)):

- If the number of \( \text{MNL}(\text{RO}) \)-sets is greater than 1, then there exists only one \( \text{MNL}(\text{RSM}) \)-set, and
  \( \text{UNION}(\text{MNM}(\text{RO}))-\text{set} \subseteq \text{UNION}(\text{MNL}(\text{RO}))-\text{set} \subseteq \text{MNL}(\text{RSM}))-\text{set} = \text{UNION}(\text{MNM}(\text{RSM}))-\text{set} = \text{UNION}(\text{MNL}(\text{RSM}))-\text{set}. \)

- If the number of \( \text{MNL}(\text{RSM}) \)-sets is greater than 1, then there exists only one \( \text{MNL}(\text{RO}) \)-set, and
  \( \text{UNION}(\text{MNM}(\text{RSM}))-\text{set} \subseteq \text{UNION}(\text{MNL}(\text{RSM}))-\text{set} \subseteq \text{MNL}(\text{RO}))-\text{set} = \text{UNION}(\text{MNM}(\text{RO}))-\text{set} = \text{UNION}(\text{MNL}(\text{RO}))-\text{set}. \)

- If there exists only one \( \text{MNL}(\text{RO}) \)-set and only one \( \text{MNL}(\text{RSM}) \)-set, then (Theorem 1 (d)) either
  \( \text{MNL}(\text{RO}))-\text{set} \subseteq \text{MNL}(\text{RSM}))-\text{set} \) or the converse is true.

**Theorem 3.** Given a digraph \((S, \succ)\), the \( \text{MNM}(\text{UNION}(\text{MNM}(CA))) \), with \( CA \in \{\text{RO}, \text{RSM}, \text{RM}\} \) axiom identifies a set, denoted by \( \text{MUMICAS}(S, \succ) \), which has the following properties:

1. It exists but is not idempotent;
2. It coincides with the optimal set (resp. strong maximal set) when the latter is non-empty and is an RM-set;
3. It is not unique because the \( \text{UNION}(\text{MNM}(CA)) \)-sets, with \( CA \in \{\text{RO}, \text{RSM}, \text{RM}\} \), can yield two or three disjoint sets with the same minimum size.

The proof is given in Section C of the Appendix (supplementary material).

This choice axiom is an alternative to maximality and \( \text{UNION}(\text{MNM}(\text{RO})) \) when the optimality identifies an empty set. Its strength stems from the fact that it is taking advantage of all minimum relative choice axioms. Its weakness is due to its non-idempotence.

### 4.4. Synthesis and Suggestions

Table 1 summarizes the results proved below on the eight introduced relative choice axioms. Thus, in a digraph context, although \( \text{MNL}(\text{RSM}) \) is attractive by its simplicity, it is not appropriate for a choice in real world applications, because it is not discriminative enough.

Other relative choice axioms are more attractive, and if we must recommend some of them, then our choices focus on:

- \( \text{UNION}(\text{MNM}(\text{RM})) \) because it coincides with maximality, it is included in the appealing candidate: \( \text{UNION}(\text{MNL}(\text{RM})) \), it characterizes one of the smallest choice set, it is idempotent, its internal structure is the set of the smallest undominated circuits of strict preference;
• MNM(UNION(MNM(CA))), with CA ∈ {RO, RSM, RM}) because it coincides with optimality and strong maximality, it is included in the other appealing candidates, it characterizes the smallest introduced relative choice set, but it is not idempotent.

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<tr>
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<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
<td>⊆ O</td>
</tr>
</tbody>
</table>

Legend: The symbol ⊆ (resp. ⊇, ⊃, ⊂) indicates the choice axiom in row is always included in (resp. contained in, intersects, equal to) the relative choice axiom in column, when both are non-empty. Whereas ⊆ indicates for some digraphs these axioms have an empty intersection. We also point out the theorem (T) or the proposition (P) of our article showing this relation. The non-proved comparisons are easily deductible from our results.

Table 1. Synthesis of set-theoretical comparisons between choice axioms.

5. Algorithmic Issues

We now consider the computational complexity of problems such as finding a MNL(CA)-set, finding a MNN(CA)-set, and counting and listing these sets. These points are crucial for adopting MNL(CA) and/or MNM(CA) as choice axioms in practice. More formally, these problems are formulated as follows.

Given a choice axiom CA ∈ {RO, RM, RSO, RSM}, we are interested in the following problem:

MNL(CA)-SET: Given a digraph G = (S, ≽), with 0 < |S| < +∞, find one MNL(CA)-set of G.

And we denote #(MNL(CA)-SET) and ENUM(MNL(CA)-SET) respectively its counting and listing versions. For this purpose, we first give (in § 5.1) a generalisation of the transitive closure concept. In the context of complete relations, Deb (1977) and Schwartz (1986) provide a characterisation of the GOCHA-set and the GETCHA-set as the maximal set of the strong transitive closure and the weak transitive closure, respectively. Therefore, we next generalise these results to relative choice axioms for CBPR. In the context of digraphs, Brandt et al. (2009) shown that, deciding whether an alternative is contained in the MNL(RSO)-set (resp. UNION(MNL(RM))-set) can be found in linear time. In fact, they proved sharper results using the following technical complexity classes for decision problems: AC⁰ and NL-completeness. We extend these results by showing the linear time computation of the MNL(CA)-SET problem and its counting and listing counterparts in § 5.2.

5.1. Characterisation of Relative Choice Sets by Transitive Closures

Transitive closures have been heavily used in the literature (see Deb, 1977; Schwartz, 1986; Suzumura, 1983; Sen (1986) among others). Given a digraph (S, ≽) with S finite...
and non-empty, and an attitude

\[ \alpha \in \mathcal{P}(\mathcal{FA}) \]

then the transitive closure of \((S, \succ)\) according to \(\alpha\) (or \(\alpha\)-transitive closure and noted \(\tau_\alpha\)) is the binary relation \(\tau_\alpha(\succ)\) defined on \(S\) in the following manner: \(\forall x, y \in S\) and \(x \neq y\),

\[
x \tau_\alpha(\succ) y \iff \exists a sequence z_1 \alpha(\succ) z_2, \ldots, z_k \alpha(\succ) z_k \text{ with } z_1 = x \text{ and } z_k = y.
\]

By construction, the transitive closures \(\tau_\alpha(\succ)\) are transitive binary relations. Accordingly, their maximal set always exists (i.e. it is non-empty) for every \((S, \succ)\) with \(S\) non-empty. Two variants appeared in SCT [Sen, 1986 § 4.1]: weak (or classical) transitive closure \(\tau_{\text{weak}}(\succ)\) and strong transitive closure \(\tau_{\text{strong}}(\succ)\). The maximal set \(M(S, \tau_\alpha(\succ))\) of \(\tau_\alpha(\succ)\) can act as a choice set for the relation \(\succ\).

The following theorem establishes a direct relationship between minimal relative sets and some \(\alpha\)-transitive closures on finite sets.

**Theorem 4.** Given a digraph \((S, \succ)\) such that \(S\) is finite and non-empty, then

(a) \(\forall \mathcal{CA} \in \{\mathcal{RO}, \mathcal{RM}, \mathcal{RSM}\}\), the \(\text{UNION}([\mathcal{MNL}([\mathcal{CA}])])\)-set of \((S, \succ)\) is equivalent to the maximal set of the \(\alpha([\mathcal{CA}])\)-transitive closure: \(\text{UNION}([\mathcal{MNL}([\mathcal{CA}])])\)-set of \((S, \succ)\) = \(\text{MAXL}(\mathcal{M})\)-set of \((S, \tau_\alpha([\mathcal{CA}])\succ)\).

(b) The \(\text{MNL}([\mathcal{RO}])\)-set of \((S, \succ)\) is equivalent to the maximal set of the \(\alpha([\mathcal{RO}])\)-transitive closure: \(\text{MNL}([\mathcal{RO}])\)-set of \((S, \succ)\) = \(\text{MAXL}(\mathcal{M})\)-set of \((S, \tau_\alpha([\mathcal{RO}])\succ)\).

**Proof:** With \(\mathcal{CA} = \mathcal{RO}\), we show that assertion (a) is true in two steps: the inclusion of the \(\text{UNION}([\mathcal{MNL}([\mathcal{CA}])])\)-set of \((S, \succ)\) in \(M(S, \tau_\alpha([\mathcal{RO}])\succ)\), followed by the converse inclusion. We denote by \(\text{UMIROS}(S, \succ)\) the \(\text{UNION}([\mathcal{MNL}([\mathcal{RO}])])\)-set of \((S, \succ)\), and by \(\succ_\alpha\) the binary relation verifying \(x \succ_\alpha y \iff x \alpha(\mathcal{RO})(\succ) y \iff \text{not}(\text{not}(y \succ x))\), and denote by \(\succ\), the binary relation verifying \(x \succ_\alpha y \iff \exists \text{ a path from } x \text{ to } y \text{ in } (S, \succ)\). So, by definition, \(\tau_\alpha(\succ) = \tau_\alpha([\mathcal{RO}])\succ\).

- We now prove the straight inclusion: every \(x \in \text{UMIROS}(S, \succ)\) belongs to \(M(S, \tau_\alpha([\mathcal{RO}])\succ)\). Let \(MRO\) be the \(\text{MNL}([\mathcal{RO}])\)-set \((\Leftrightarrow MRO \subseteq \text{UMIROS}(S, \succ))\) such that \(x \in MRO\). Then:

  (i) \(\forall y \in MRO \setminus \{x\}\), there exists a path from \(x\) to \(y\) and a path from \(y\) to \(x\) in \((S, \succ_\alpha)\) (i.e. \(MRO\) is strongly connected: Proposition 6 with \(\mathcal{CA} = \mathcal{RO}\) \(\Leftrightarrow x \equiv_1 y\) (by definition of \(\equiv_1\)) \(\Rightarrow \text{not}(y \equiv_{\tau_\alpha} x)\).

  (ii) \(\forall y \in S \setminus MRO, x \succ y\) (because \(MRO\) is an \(\mathcal{RO}\)-set) \(\Leftrightarrow \text{not}(\text{not}(y \succ_\alpha x))\) (by definition of \(\succ_\alpha\)). More generally, \(\forall (y, z) \in (S \setminus MRO) \times MRO, z \succ y \iff \text{not}(y \succ_\alpha z) \iff \text{the cutset } \Omega(MRO) = \emptyset \text{ in } (S, \succ_\alpha) \iff \text{there exists no path from } y \text{ to } z \text{ in } (S, \succ_\alpha) \iff \text{not}(y \equiv_{\tau_\alpha} z)\). In particular, \(\text{not}(y \equiv_{\tau_\alpha} x)\) and then implies: \(\text{not}(y \equiv_{\tau_\alpha} x)\).

Finally, every \(x \in \text{UMIROS}(S, \succ)\) verifies: \(\forall y \in S \setminus \{x\}\), \(\text{not}(y \equiv_{\tau_\alpha} x) \iff x \in M(S, \equiv_{\tau_\alpha})\). In other words, \(\text{UMIROS}(S, \succ) \subseteq M(S, \equiv_{\tau_\alpha})\).

- We now show the converse inclusion through reasoning by contradiction. Suppose there exists \(y \in M(S, \equiv_{\tau_\alpha}) \setminus \text{UMIROS}(S, \succ)\). Then, \(\forall z \in S \setminus \{y\}\), \(\text{not}(z \equiv_{\tau_\alpha} y)\). Yet, \(\forall x \in \text{UMIROS}(S, \succ)\), \(x \equiv_{\tau_\alpha} y\) (because the union of \(\mathcal{RO}\)-sets \(\text{UMIROS}(S, \succ)\) is an \(\mathcal{RO}\)-set according to Proposition 2) \(\iff \text{not}(y \equiv_{\alpha} x)\) (by definition of \(\equiv_{\alpha}\)) \(\iff \text{the cutset } \Omega(\text{UMIROS}(S, \succ)) = \emptyset \text{ in } (S, \equiv_{\alpha}) \iff \text{there exists no path from } y \text{ to } x \text{ in } (S, \equiv_{\alpha}) \iff \text{not}(y \equiv_{\tau_\alpha} x)\). This contradicts the initial assumption: \(\forall x \in S \setminus \{y\}, \text{not}(x \equiv_{\tau_\alpha} y)\). Therefore, there exists no \(y \in M(S, \equiv_{\tau_\alpha}) \setminus \text{UMIROS}(S, \succ) \iff M(S, \equiv_{\tau_\alpha}) \subseteq \text{UMIROS}(S, \succ)\).

And finally, \(M(S, \equiv_{\tau_\alpha}) = \text{UMIROS}(S, \succ)\).

An equivalent proof for the choice axioms \(\mathcal{RM}\) and \(\mathcal{RSM}\), and a simplified one for assertion (b), can be made as well. \(\Box\)

---

9 See Definition 1.
We have seen above (Proposition 6) that a strong relationship links minimal relative axioms and strong connectivity. The following result goes further toward describing the maximal sets of transitive closures by MNL(CA), corroborating their attractiveness.

**Proposition 12.** \( \forall CA \in \{RO, RM, RSM\} \), there exists a bijection between strongly connected components of \( M(S, \tau_{(CA)}(\succeq)) \) and the MNL(CA)-sets.

**Proof:** This follows rather straightforwardly from Proposition 6 and Theorem 4. \( \square \)

Minimal relative choice sets describe the internal structure (see § 3.3) of maximal sets of transitive closures and restitute them advantageously. Thus, the formers make an “axiomatic decomposition” of the latters into subsets corresponding to their maximal equivalence classes: the maximal strongly connected components.

### 5.2. Linear Time Computation of Minimal Relative Choice Sets

We now focus on the computational complexity of the three problems: MINIMAL(CA)-SET, \#(MINIMAL(CA)-SET) and ENUM(MINIMAL(CA)-SET). Consider the following polynomial time problem.

**STRONGLY CONNECTED COMPONENTS:** Given a digraph \( G = (S, \succeq) \), with \( 0 < |S| < +\infty \), list the strongly connected components of \( G \).

This problem can be efficiently solved, for example, using Tarjan’s algorithm (Bang-Jensen and Gutin, 2001) with a \( O(|S| + |\succeq|) \) worst case time complexity. Consequently, we have the following theorem:

**Theorem 5.** \( \forall CA \in \{RO, RM, RSO, RSM\} \), the problems MINIMAL(CA)-SET, \#(MINIMAL(CA)-SET) and ENUM(MINIMAL(CA)-SET) can be solved in linear time.

**Proof:** Given a choice axiom \( CA \in \{RO, RM, RSO, RSM\} \) and a digraph \( (S, \succeq) \), one algorithm is described below: (1) Design the digraph \( (S, \alpha(CA)(\succeq)) \) from \( (S, \succeq) \); (2) Find the reduced digraph (acyclic digraph of strongly connected components) associated with \( (S, \alpha(CA)(\succeq)) \); (3) Find the maximal set of the reduced digraph. Each maximal vertex of the reduced graph identifies one maximal strongly connected component of \( (S, \alpha(CA)(\succeq)) \), that is a MNL(CA)-set of \( (S, \succeq) \) according to Proposition 10 and Theorem 4. Therefore, this algorithm solves MINIMAL(CA)-SET if we stop it from the first designed maximal strongly connected component. It solves ENUM(MINIMAL(CA)-SET) if we leave the algorithm building all these components. Finally, we can easily adapt the algorithm to count these components. Remark at last that the number of MNL(CA)-sets is always smaller than or equal to the cardinality of \( S \) for any CBPR. The reason is simple: the MNL(CA)-sets are disjoint (Proposition 5\( a \)) and their union is a subset of \( S \). \( \square \)

The above algorithm can easily be adapted to find a MNM(CA)-set, and to list and count them.

### 5.3. Practical Application

We use these algorithms on a real-world best choice problem (for surveys on the applications, see chapter 4 of Figueira et al. (2005) and Roy and Bouyssou (1993)). Therefore, we consider the problem of choosing a postal parcels sorting machine thoroughly discussed by Roy and Bouyssou (1993, pp 501-541). We observe a set \( S \) of 9 potential installations evaluated on the coherent family of 12 criteria. The
implementation of the multiple criteria aggregation procedure (MCAP) of ELECTRE IS reaches the global outranking relation \( \succeq_1 \) given in Fig. 9 (a).

![Fig. 9. Examples of digraphs.](image)

(a): Digraph \( G_1 = (S, \succeq_1) \)

(b): Digraph \( G_2 = (S, \succeq_2) \)

(c): Digraph \( G_3 = (S, \succeq_3) \)

Table 2 (columns two and sixth) summarizes the computed choice sets from the diverse targeted choice axioms on \((S, \succeq_1)\). The ELECTRE IS method uses only kernels, defined as follows (Berge, 1973):

Given a digraph \((S, \succeq)\), the subset \(A \subseteq S\) is a (dominant) kernel iff:

\[
\forall y \in S \setminus A, \exists x \in A \text{ such that } x \succeq y. \quad \text{(von Neumann-Morgenstern’s domination)} \tag{10}
\]

\[
\forall x, y \in A \text{ and } x \neq y, \neg(x \succeq y). \quad \text{(internal stability)} \tag{11}
\]

Applied on digraph \((S, \succeq_1)\), the kernel is not unique: \(\{1,2,7,9\}\) and \(\{1,3,7,9\}\); both are greater than the strong maximal set \(\{1,7,9\}\) and than the minimum among the unions of minimum relative choice sets \(\{1,7,9\}\) introduced in § 4.3. Roy and Bouyssou have produced a complementary decision aid analysis that also obtains our result. This application points out the relevance of relative choice axioms even when kernels and some usual choice axioms exist. Next, Roy and Bouyssou made a robustness analysis of the choice set, justified by the difficulties in setting parameters of the MCAP of ELECTRE IS. They should consider 81 preference relations on \(S\); but lack of time (an imminent deadline) for an interactive support and irrelevance of some of these relations\(^{10}\) motivated the authors to consider only 20. Although they have not provided them, they indicated these relations are close to \(\succeq_1\).

Table 2. Kernels, usual and relative choice sets of the 3 digraphs in Fig. 9.

<table>
<thead>
<tr>
<th>Axioms</th>
<th>Choice sets of digraphs: ((S, \succeq_1))</th>
<th>Choice sets of digraphs: ((S, \succeq_2))</th>
<th>Choice sets of digraphs: ((S, \succeq_3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{MNL}(\text{SO}))</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(\text{MNL}(\text{O}))</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(\text{MNL}(\text{SM}))</td>
<td>({1,7,9})</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(\text{MNL}(\text{M}))</td>
<td>({1,2,3,7,9})</td>
<td>({2,3})</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>Kernels</td>
<td>({1,7,9})</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
</tbody>
</table>

Consequently, we now consider two relations allowing no particular properties such as P-acyclicity (see Fig. 9 (b) and (c)):

- The relation \(\succ_2\) on \(S\) derived from \(\succeq_1\) by considering pairwise comparisons: \(x_1 \succ_2 x_9 \succ_2 x_7 \succ_2 x_1\);
- The relation \(\succ_3\) on \(S\) derived from \(\succeq_2\) by considering pairwise comparisons: \(x_2 \succ_3 x_3 \succ_3 x_4 \succ_3 x_2\) and \(x_5 \simeq_3 x_9\);

These relations are not relevant for kernel, strong optimality or optimality which identify an empty set (Table 2). Whereas \(\text{MNL}(\text{RO})\) and \(\text{MNL}(\text{RSO})\) do not allow to

\(^{10}\) We suppose some relations are irrelevant because the kernel axiom is unable to select a non-empty choice set.
discriminate a proper subset of $S$ for the choice set. Only the other axioms enable significant choice sets to emerge:

For relation $\succ_2$, the non-empty maximal set $\{2, 3\} = \text{UNION}(\text{MNM}(\text{RM}))-\text{set}$ (Theorem 2), and the maximal set of the classical transitive closure (given here by the single $\text{MNL}(\text{RSM})$)-set) coincides with the initial choice set $\{1, 7, 9\}$, which is included in the $\text{UNION}(\text{MNL}(\text{RM}))-\text{set}$.

For relation $\succ_3$, the maximal set is empty and the $\text{MNL}(\text{RSM})$-set only removes alternative 8. The $\text{MNL}(\text{RM})$-set $\{1, 7, 9\}$ discriminates the set $S$ in a more adequate way, and identifies a strict preference circuit (a property of a $\text{MNL}(\text{RM})$-set: Proposition 6).

Despite the lack of information (a list of 81 digraphs), we reach most of the authors’ conclusions: Alternatives 5, 6 and 8 are elements of no relevant choice sets, whereas 2 and 3 are sometimes chosen, and 1, 7 and 9 are more frequently, and even almost always, chosen. Only alternative 4 has not been plainly ranked with alternatives 2 and 3.

6. Conclusion and Perspectives

In this paper, we generalized several choice axioms (top-cycle, GETCHA and GOCHA) of axiomatic choice theory from tournaments and weak tournaments to reflexive digraphs. We denoted these as relative choice axioms. The strength of these axioms stems from their ability to identify a relevant non-empty choice set everywhere outside the domain of existence of usual choice axioms in MCDA (optimality, maximality and kernels), and from their ability to coincide with optimality or maximality inside this domain of existence. Also, and definitely not least of their properties, relative choice sets can be computed in a linear time given an initial digraph.

Beside their propensities to replace the usual choice axioms in MCDA, relative choice axioms should also significantly improve the robustness analysis (Roy, 1998, 2010) supplementing the best choice problem in multiple criteria aiding procedures. This is an interesting first direction of future research. Another one is the generalisation of relative choice axioms to other preference structures. For the case when the dissociation between tentative incompleteness and assertive incompleteness (Sen, 1997, 2002) is allowed, how does this family grow larger? What are the preserved and new properties? Another example comes from the permission of other asymmetric fundamental attitudes as weak preferences and weak aversions (Roy and Bouyssou, 1993). More broadly, how does this family change when fuzzy or valued binary preference relations (Figueira et al., 2005) are used?

Acknowledgements

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Appendix

Section A: Proof of Proposition 6

Before showing this proposition, we first provide some additional definitions. Given a digraph $G = (S, \succ)$ and a set $U \subset S$, we consider the following arc sets:

- $\Omega^+(U) = \{(x,y) \in \succ \mid x \in U \text{ and } y \notin U\}$
- $\Omega^-(U) = \{(x,y) \in \succ \mid x \notin U \text{ and } y \in U\}$

The cutset (or cocycle) relative to $U$ in $G$ is the arc set $\Omega(U) = \Omega^+(U) \cup \Omega^-(U)$. If $\Omega^+(U) = \emptyset$ or $\Omega^-(U) = \emptyset$ then the cocycle $\Omega(U)$ is called a cocircuit.

The well-known following lemma (see, for example Berge (1973, chap. 2)) recalls the link between path, cocircuit and strong connectivity in a digraph.

**Lemma 1 (corollary to Minty’s lemma).** Given a finite connected digraph $G$ containing at least one arc, the following conditions are equivalent: (i) $G$ is strongly connected, (ii) Every arc is an element of a circuit, and (iii) $G$ contains no cocircuit.

**Proof (Proposition 6):** In fact the proposition is a disjunction of conclusions: In assertion (c), if the constraint of cardinality ($|\text{CAS}| \geq 3$) is replaced by $|\text{CAS}| = 1$ [resp. $|\text{CAS}| = 2$], then we obtain the assertion (a) [resp. (b)]. Assertions (a) and (b) are easy to prove. We therefore focus on assertion (c). We give a proof in the general case when $\text{CAS} = \text{RO}$ (and the proof for $\text{CAS} \in \{\text{RM, RSO, RSM}\}$ is analogous). Consider the digraph $(\text{CAS}, \succ_1)$ obtained from $(\text{CAS}, \succ)$ by:

$$ z \succ_1 t \iff z \alpha(\text{RO})(\succ) t \iff \text{not}(t \succ z), \; \forall (z,t) \in \text{CAS}. \quad (12) $$

We now show that if $\text{CAS}$ is a $\text{MNL(RO)}$-set, then the digraph $(\text{CAS}, \succ_1)$ is strongly connected. We reason by contradiction. Two cases are possible: either $(\text{CAS}, \succ_1)$ is connected but not strongly connected, or it is not connected. We consider each case separately:

- Suppose that $(\text{CAS}, \succ_1)$ is not connected. Then there exists a bipartition $(M_1, M_2)$ of $\text{CAS}$ such that there exists no arc between $M_1$ and $M_2$. Consequently, $\forall (z,t) \in M_1 \times M_2$, $\text{not}(z \succ_1 t)$ and $\text{not}(t \succ_1 z) \iff z \simeq t$ (by using definition (12)). This means that $M_1$ and $M_2$ are two disjoint RO-sets of $(S, \succ)$ because $\forall i \in \{1, 2\}$ and $\forall (z,t) \in M_i \times (S \setminus M_i)$, $z \succ t$. But this deduction contradicts that $\text{CAS}$ is a $\text{MNL(RO)}$-set.

- Suppose now that $(\text{CAS}, \succ_1)$ is connected, but not strongly connected. This is equivalent to say (Lemma 1) that there exists a non-empty set $U \subset \text{CAS}$ such that $\Omega(U)$ is a cocircuit in $(\text{CAS}, \succ_1)$. By using formula (12), the existence of a cocircuit $\Omega(U)$ in the digraph $(\text{CAS}, \succ_1)$ is equivalent to:
  - In the case when $\Omega^+(U) = \emptyset$, then $\forall (z,t) \in U \times (\text{CAS} \setminus U)$, $\text{not}(z \succ_1 t) \iff t \succ z$.
  - Otherwise $\Omega^-(U) = \emptyset$, and $U$ is an RO-set of $(S, \succ)$. Another contradiction.

In summary, all these possibilities result in a contradiction. Consequently, if $\text{CAS}$ is a $\text{MNL(RO)}$-set, then assertion (c) is true.

Section B: Proof of Theorem 1

In order to show this result, we introduce first the following lemma:
Lemma 2. Given a digraph \((S, \succ)
\) such that \(S\) is finite and non-empty, then:

(a) The intersection between a \(\text{MNL(RO)}\)-set \(\text{MRO}\) and a \(\text{MNL(RSM)}\)-set \(\text{MRSM}\) is non-empty \((\Leftrightarrow \text{MRO} \cap \text{MRSM} \neq \emptyset)\). Moreover, this intersection is an \(\text{RM}\)-set. Hence it contains at least one \(\text{MNL(RM)}\)-set.

(b) The union of any \(\text{MNL(RO)}\)-set \(\text{MRO}\) and any \(\text{MNL(RSM)}\)-set \(\text{MRSM}\) contains the union of \(\text{MNL(RM)}\)-sets \((\Leftrightarrow \cup \text{union(\text{MNL(RM))})-set} \subseteq \text{MRO} \cup \text{MRSM}\).

Proof: Assertion (a), we first show that the intersection of any \(\text{MNL(RO)}\)-set \(\text{MRO}\) and any \(\text{MNL(RSM)}\)-set \(\text{MRSM}\) is never empty. We suppose \(\text{MRO} \cap \text{MRSM} = \emptyset\). Accordingly, \(\forall x \in \text{MRO} \land \forall y \in \text{MRSM}, x \not\succ y\) (by definition of \(\text{MRO}\)) and \(\not\text{not}(x \succ y)\) (by definition of \(\text{MRSM}\). Since this is contradictory, we therefore have \(\text{MRO} \cap \text{MRSM} \neq \emptyset\).

Next, such an intersection is an \(\text{RM}\)-set, because \(\text{MRO}\) and \(\text{MRSM}\) are \(\text{RM}\)-sets (Proposition 2). Finally, it is obvious that \(\text{MRO} \cap \text{MRSM}\) contains a \(\text{MNL(RM)}\)-set.

Assertion (b), we first show that \(\text{union(\text{MNL(RM))}-set} \subseteq \text{MRO} \cup \text{MRSM}\). We suppose there exists a \(\text{MNL(RM)}\)-set \(\text{MRM}\) disjoint from \(\text{MRO} \cup \text{MRSM}\). Recall that they cannot intersect because \(\text{MRO} \cup \text{MRSM}\) is an \(\text{RM}\)-set (Proposition 2) and \(\text{MRM}\) is minimal w.r.t. inclusion. Consequently, \(\forall x \in \text{MRM} \land \forall y \in \text{MRSM}, \not\text{not}(y \succ x)\) and \(\not\text{not}(\not\text{not}(x \succ y))\) ⇔ \(x \not\approx y \lor | x \parallel y\). In the particular case of \(x \in \text{MRO} \lor \text{MRSM}\) (which is always non-empty according to assertion (a)), we have: \(\not\text{not}(x \succ y)\) (definition of \(\text{MRM}\)) and \(x \not\succ y\) (because \(x \in \text{MRO}\) and \(y \not\in S \setminus \text{MRO}\)). This is a contradiction. Consequently, every \(\text{MNL(RM)}\)-set \(\text{MRM}\) is included in \(\text{MRO} \lor \text{MRSM} \Leftrightarrow \text{union(\text{MNL(RM))}-set} \subseteq \text{MRO} \lor \text{MRSM}\). □

Using this lemma, we now give a proof of Theorem 1.

Proof (Theorem 1): We suppose that \((\text{CA}_1, \text{CA}_2) = (\text{RO, RSM})\). Because of the duality between \(\text{RSM}\) and \(\text{RO}\), an equivalent proof is available for the other case as well.

We first prove assertion (b). We consider two \(\text{MNL(RO)}\)-sets \(\text{MRO}_1\) and \(\text{MRO}_2\), and a \(\text{MNL(RSM)}\)-set \(\text{MRSM}\). According to Proposition 5 (b), we have: \(\forall (x, y) \in \text{MRO}_1 \times \text{MRO}_2, x \not\approx y\). Moreover, according to Lemma 2 (a), there exists \(x_1 \in \text{MRO}_1 \cap \text{MRSM}\). If there exists \(y_1 \in \text{MRO}_2 \setminus \text{MRSM}\), then \(x_1 \not\approx y_1\). But, as \(\text{MRSM}\) is an \(\text{RM}\)-set, then (via formula (8)) \(\not\text{not}(y_1 \succ x_1)\) or \(\not\text{not}(x_1 \succ y_1)\). This is contradictory. Therefore, \(\text{MRO}_2 \subseteq \text{MRSM}\). In the same manner, \(\text{MRO}_1 \subseteq \text{MRSM}\). Finally, if the number of \(\text{MNL(RO)}\)-sets is greater than 1, then every \(\text{MNL(RO)}\)-set is included in any \(\text{MNL(RSM)}\)-set.

Assertion (a) is a direct consequence of assertion (b) and Proposition 5 (a) for \(\text{MNL(RSM)}\)-sets.

Assertion (c) is a direct consequence of assertion (a) and Lemma 2 (b).

Assertion (d): We denote \(\text{MRO}\) the only \(\text{MNL(RO)}\)-set and \(\text{MRSM}\) the only \(\text{MNL(RSM)}\)-set. Then, either \(\text{MRO} \subseteq \text{MRSM}\) or the converse is true. Indeed, reasoning by contradiction: If there is no inclusion, there must exist at the same time at least one \(x \in \text{MRO} \setminus \text{MRSM}\) and at least one \(y \in \text{MRSM} \setminus \text{MRO}\). Formulas (5) and (7) then establish that \(x \not\succ y\) and \(\not\text{not}(x \not\succ y)\). This is a contradiction, therefore either \(\text{MRO} \subseteq \text{MRSM}\), or \(\text{MRSM} \subseteq \text{MRO}\). □

Section C: Proof of Theorem 3

Proof: For (1), the non-emptiness is warranted by the existence of the \(\text{MNL(CA)}\)-sets, with \(\text{CA} \in \{\text{RO, RSM, RM}\}\) (Proposition 2), otherwise the alternative set \(S\) is empty. We show by an example that this axiom is not idempotent. Consider a digraph \(G = (S, \succ)\), illustrated in Fig. 9, partitioned into 3 sub-digraphs made up of the respective vertices \(X, \ldots\)
Y, Z, with \( X \cap Y = \emptyset, X \cap Z = \emptyset, Y \cap Z = \emptyset \) and \( X \cup Y \cup Z = S \). These 3 sub-digraphs are themselves made up as follows:

- \( (X, \succ) \) is partitioned into 2 elementary circuits of 6 vertices: \( C_{X1} \) and \( C_{X2} \).
- \( (Y, \succ) \) is partitioned into 3 circuits of 3 vertices: \( C_{Y1}, C_{Y2} \) and \( C_{Y3} \).
- \( (Z, \succ) \) is partitioned into 2 circuits of 3 vertices: \( C_{Z1} \) and \( C_{Z2} \).

For every couple of circuits \( (C_{\alpha}, C_{\beta}) \) of two different sub-digraphs, every vertex of \( C_{\alpha} \) is indifferent with every vertex of \( C_{\beta} \). Also, every vertex of \( C_{X1} \) is indifferent with every vertex of \( C_{X2} \). At last, for every couple of different circuits \( (C_{\alpha}, C_{\beta}) \) in the same sub-digraph \( (Y, \succ) \) or \( (Z, \succ) \), every vertex of \( C_{\alpha} \) is incomparable with every vertex of \( C_{\beta} \).

Fig. 9. Illustration of the non-idempotence of the combination of minimum choice axioms. 4

The \( \text{UNION}(\text{MNM}(\text{RM}))-\text{set} = Y \cup Z \), while the \( \text{UNION}(\text{MNM}(\text{RO}))-\text{set} = X \cup Z \), and the single \( \text{MNL}(\text{RSM})\)-set = \( S \). Accordingly, \( \text{MUMICAS}(S, \succ) = Y \cup Z = \text{the UNION}(\text{MNM}(\text{RM}))-\text{set} \).

Next, we compute \( \text{MUMICAS}(\text{MUMICAS}(S, \succ), \succ) \). The \( \text{UNION}(\text{MNM}(\text{RM}))-\text{set} \) of \( \text{MUMICAS}(S, \succ) \) is \( Y \cup Z \), while the single \( \text{MNM}(\text{RO})\)-set = \( Z \), and the single \( \text{MNL}(\text{RSM})\)-set = \( \text{MUMICAS}(S, \succ) \). Consequently, \( \text{MUMICAS}(\text{MUMICAS}(S, \succ), \succ) = Z \subseteq Y \cup Z = \text{MUMICAS}(S, \succ) \). Finally, the \( \text{MNM}(\text{UNION}(\text{MNM}(\text{CA})), \text{with CA} \in \{\text{RO, RSM, RM}\}) \) axiom is not idempotent.

(2) The \( \text{MUMICAS}(S, \succ) \) is always an \( \text{RM}\)-set because of Proposition 2. Moreover, this new choice set coincides with the optimal set when the latter is non-empty. Indeed, when the optimal set is non-empty, it coincides with the \( \text{UNION}(\text{MNM}(\text{RO}))-\text{set} \) (Theorem 2); this latter set is included into the \( \text{UNION}(\text{MNM}(\text{RSM}))-\text{set} \) (Theorem 1), and it is included into the maximal set (Proposition 1 (d)), the latter coinciding with the \( \text{UNION}(\text{MNM}(\text{RM}))-\text{set} \) (Theorem 2). A similar proof exists for the strong maximal set.

(3) It is straightforward to identify digraphs proving this assertion.

References


White DJ. Kernels of preferences structures. Econometrica 1977; 45 (1); 91-100.